July 6, 2014

Sector Models – A Toolkit for Teaching General Relativity. Part 1: Curved Spaces and Spacetimes

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Abstract.
Teaching the general theory of relativity to high school or undergraduate students must be based on an approach that is conceptual rather than mathematical. In this paper we present such an approach that requires no more than elementary mathematics. The central idea of this introduction to general relativity is the use of so-called sector models. Sector models describe curved spaces the Regge calculus way by subdivision into blocks with euclidean geometry. This procedure is similar to the approximation of a curved surface by flat triangles. We outline a workshop for high school and undergraduate students that introduces the notion of curved space by means of sector models of black holes. We further describe the extension to sector models of curved spacetimes. The spacetime models are suitable for learners with a basic knowledge of special relativity. The teaching materials presented in this paper are available online for teaching purposes, see http://www.spacetimetravel.org.
(Some figures are in colour only in the electronic version.)

1. Introduction

The general theory of relativity is one of the two fundamental advancements in physics in the 20th century, the other one being quantum theory. Tested to high accuracy within the solar system, the theory is well established. It belongs to the foundations of today's physical view of the world; for the understanding of many astrophysical phenomena it is of major importance. Also, the theory of relativity strongly attracts the interest of the general public, not least because of its relevance for understanding exotic phenomena like black holes and for cosmological questions about the beginning and the end of the universe.

However, teaching this theory that is so important and so fascinating to many students is faced with a basic problem at the high school and undergraduate levels. The standard introduction provides the necessary mathematical tools, moves on to the motivation of the field equations and to the derivation of analytic solutions, and then examines the solutions, especially with regard to the paths of particles and light. The mathematical tools that this programme is based on are extensive and way beyond the elementary mathematics taught in school. For high school and beginning undergraduate students the standard approach is therefore not accessible. It is likewise out of reach of a physics minor programme since there is not enough time to develop the mathematical background and then continue all the way to the astrophysical phenomena.

Consequently, it is a desideratum to teach general relativity in a way that is based on elementary mathematics only. This objective, already stated by Einstein in 1916 (Einstein, 1916) has been pursued in many different ways both in the development of teaching materials and in popular science publications. The following four approaches are widely-used:

1. Conclusions from the principle of equivalence: Gravitational light deflection and time dilation are introduced by means of thought experiments based on the equivalence principle (e.g. in Einstein, 1916; Gamow, 1961; Sartori, 1996; Tipler and Mosca, 2008; Stannard, 2008).

2. Description of the geometry of curved surfaces: Using simple curved surfaces, e.g. the surface of a sphere, geometric concepts that are important for curved spacetimes are introduced, for instance metric, geodesics and curvature (e.g. in Sartori, 1996; Hartle, 2003; Stannard, 2008; Natário, 2011).

3. Computations based on Newtonian dynamics: Newtonian computations are performed for phenomena that should in fact be described in terms of general relativity, like gravitational light deflection, black holes and the cosmological expansion (e.g. in Ehlers and Pössel, 2003; Lotze, 2005). The idea behind this is to use familiar concepts like force and energy in order to head straight for the phenomena (e.g. light deflection, described as deflection of a classical particle stream) and on to astrophysical applications (e.g. gravitational lenses). The conceptual change associated with general relativity is an issue that is not raised in this approach.

4. Analogies: Especially in popular science publications analogies are widely used

both to introduce basic concepts (e.g. a ball making a depression in a rubber sheet to illustrate the concept "mass curves space") and to describe relativistic phenomena (e.g.: a wine glass base acting as a "gravitational lens") (e.g. Price and Grover, 2001; Lotze, 2004). The use of analogies entails a considerable risk of the formation of misconceptions (Zahn and Kraus, 2010).

Another pictorial but more sophisticated description of curved spaces and spacetimes uses embedding diagrams. Embeddings of two-dimensional subspaces in three-dimensional flat space have been described both for the spatial (Flamm, 1916; Epstein, 1994; Jonsson, 2001, 2005) and for the spatiotemporal case (Marolf, 1998). The significance of embeddings goes beyond that of analogies as mentioned in items 2 and 4 above. Embeddings represent subspaces of definite spacetimes so that their geometric properties have physical significance. Like the analogies mentioned above, embeddings are limited to two-dimensional subspaces.

This paper presents a novel approach to teaching general relativity that like the above-mentioned ones uses elementary mathematics at most. Its objective is to convey the basic ideas of the general theory of relativity as summarized succinctly by John Wheeler in his well-known saying (Wheeler, 1990):

> *Spacetime tells matter how to move.*
> *Matter tells spacetime how to curve.*

This summary shows that the following three fundamental questions should be addressed:

(i) What is a curved spacetime?
(ii) How does matter move in a curved spacetime?
(iii) How is the curvature of the spacetime linked to the distribution of matter?

The central idea in the approach presented in this paper is the use of so-called *sector models*. A sector model depicts a two- or three-dimensional subspace of a curved spacetime. It is true to scale and is built according to the description of spacetimes in the Regge calculus by way of subdivision into uncurved blocks (Regge, 1961). In case of a two-dimensional space the blocks are flat elements of area. In case of a three-dimensional space the blocks are bricks, the geometry within each brick being euclidean. Spatiotemporal subspaces are represented by sectors that have internal Minkowski geometry. To build a physical model, the sectors can be realized as pieces of paper or as boxes made from cardboard.

Using sector models the three questions stated above can be treated in a descriptive way. The approach is suitable for undergraduate students and can also be used with advanced high school students. Since this description of the basic ideas is closely connected with the standard mathematical presentation, the approach can also be used as a complement to a standard textbook in order to promote a geometric understanding of curved spacetimes.

This paper addresses the first of the three fundamental questions in more detail and shows how the properties of curved space can be conveyed using sector models. The Schwarzschild spacetime of a black hole will serve as an example for this introduction. Section 2 outlines a workshop introducing the notion of curved space. The workshop has been held several times for high-school and for undergraduate students. The concept of the sector model is introduced in this section in a non-technical way. Details on the computation of the sector models and on their properties are given in section 3, a spacetime sector model is presented in section 4, and section 5 gives an outlook on further applications of sector models.

2. A do-it-yourself black hole

This workshop introduces the three-dimensional curved space of a black hole. The concept of a curved space is presented in a descriptive way in order to promote a qualitative understanding of the properties of curved spaces. The black hole is a convenient example for two reasons: As an exotic object with frequent media coverage it appeals to many people. Also, close to a black hole relativistic effects are sufficiently large to be clearly visible in a true-to-scale model.

2.1. Curved surfaces

For a start we introduce the concept of curvature using curved surfaces by way of example. In doing so we distinguish positive, negative and zero curvature. The sphere, the saddle and the plane are introduced as prototypes of the three types of curved surfaces.

We state the criterion that permits to determine the type of curvature: A small piece of the surface is cut out and flattened on a plane. If it tears when flattened, the curvature is positive, if it buckles, the curvature is negative. A piece that can be flattened without tearing or buckling has zero curvature. This criterion determines the sign of the internal (Gaussian) curvature. For practice it is applied to different surfaces. Useful examples are in particular the torus (negative curvature at the inner rim, positive curvature at the outer rim) to demonstrate that curvature can vary from point to point, and the cylinder (zero curvature) to show that curvature as defined here is not in perfect agreement with the everyday use of the word.

The second step illustrates how a curved surface can be approximated by small flat pieces. Using glue and the cut-out sheets shown in figure 1 one group is given the task to build two surfaces (figure 2(a), 2(b)) and to determine the sign of the curvature in each case. A second group is instructed to just cut out the flat pieces without glueing them together, to lay them out on the table (figure 2(c), 2(d)) and to find out the sign of the curvature from this set-up. The criterion "tearing" or "buckling" can easily be applied in this case also (figure 2(e), 2(f)).

The aim of the third step is to accept the representation of a curved surface by pieces

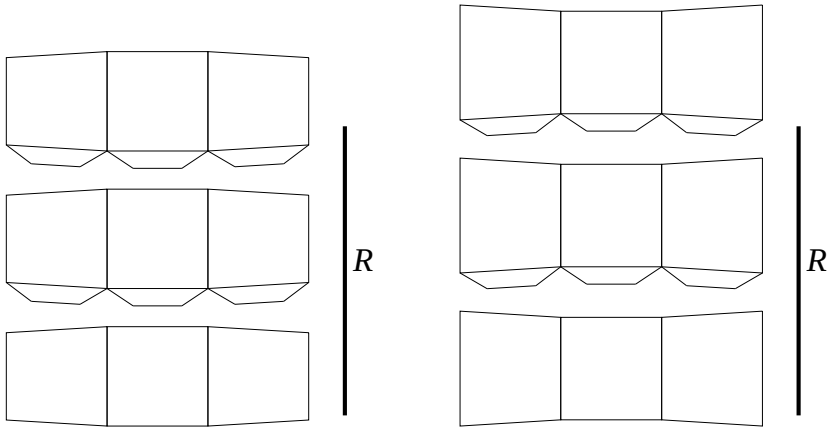


Figure 1. Cut-out sheets for two curved surfaces. Left: spherical cap with positive curvature $K = 1/R^2$, R being the radius of the sphere. Right: Saddle with constant negative curvature $K = -1/R^2$. The length of the scale bar indicates the value of R.

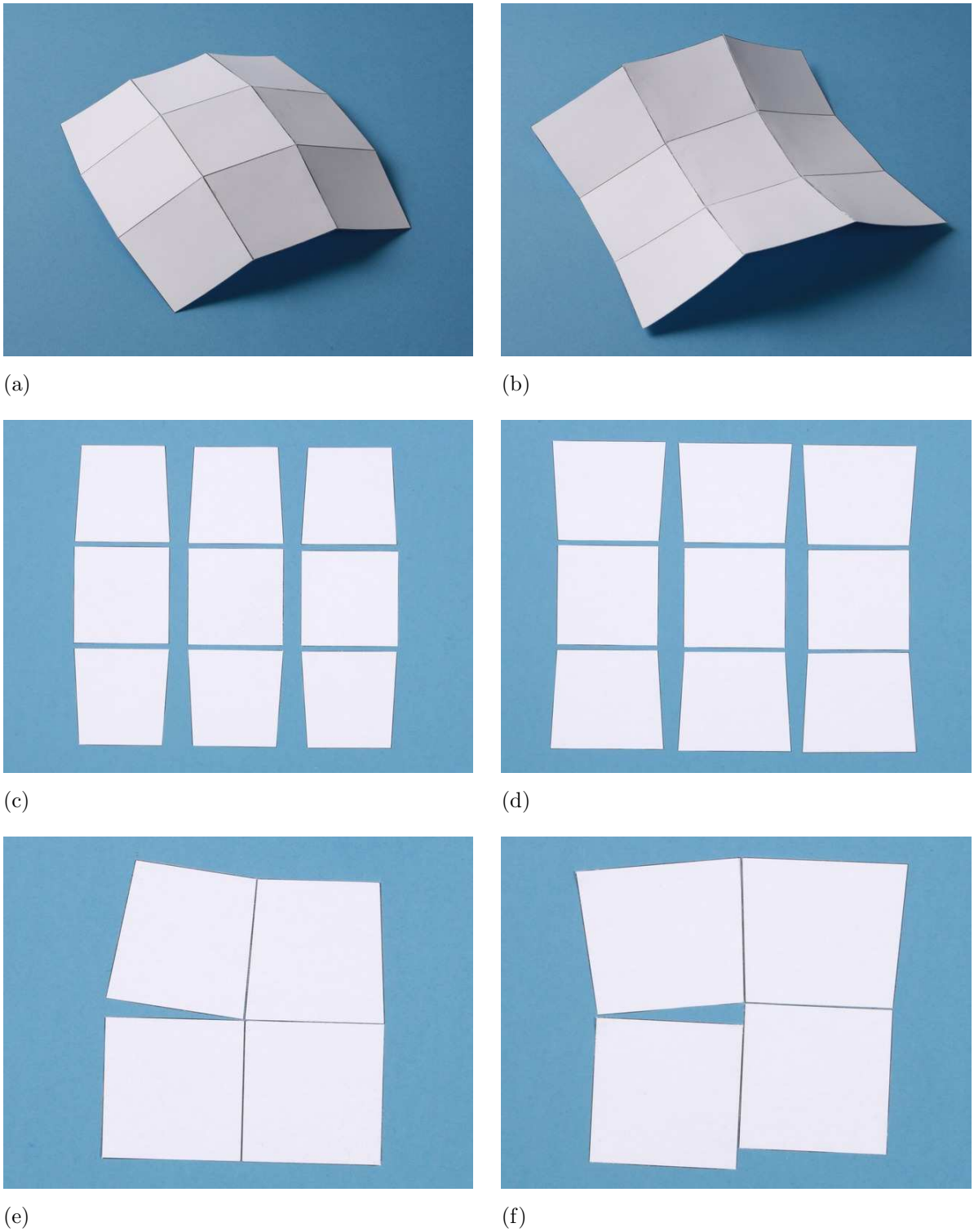
laid out in the plane (figure 2(c), 2(d)) as a useful and indeed equivalent alternative to the more familiar picture of the surface bending into the third dimension (figure 2(a), 2(b)). To prepare for this new point of view we describe the world view of "flatlanders", the inhabitants of "Flatland" in Edwin Abbott's tale of the same name (Abbott, 1884):

I call our world Flatland (...). Image a vast sheet of paper on which (...) figures (...) move freely about, but without the power of rising above or sinking below it, very much like shadows (...).

The flatlanders move in two dimensions (forward – backward, right – left), the third dimension (up – down) is not only unaccessible to them, but is beyond their imagination. When we extend Abbot's flatland to curved surfaces, the flatlanders still move only within the surface forward – backward and right – left. Lacking the concept of up and down, they cannot conceive of a surface bending into an embedding three-dimensional space. Nevertheless, they are able to study the curvature of their world. To do this, they fabricate a model similar to the ones shown in figures 2(c), 2(d): A tract is subdivided into lots that are small enough to be approximately flat. The lengths of the edges of all the lots are measured, the pieces reproduced on a reduced scale and laid out in the plane (similar to the arrangement in figures 2(c) and 2(d)). The pieces fit together without gaps only if the region in question is flat. Otherwise, the region has curvature the sign of which can be found out by applying the criterion "tearing or buckling?" at the vertices (similar to the arrangement in figures 2(e) and 2(f)).

2.2. Curved space

In step four we now study a three-dimensional curved space. Being "spacelanders" familiar with three dimensions but unable to conceive of a higher dimensional space, we can examine the curvature of our three-dimensional space the same way that flatlanders examine curved surfaces: We subdivide a spatial region into pieces (three-dimensional blocks) that are small enough to be approximately euclidean. The lengths of all the



(a) (b)

(c) (d)

(e) (f)

Figure 2. Surfaces with positive curvature (a) and negative curvature (b) approximated by flat pieces. Their sector models (c, d) indicate the positive curvature by "tearing" (e) and the negative curvature by "buckling" (f), respectively.

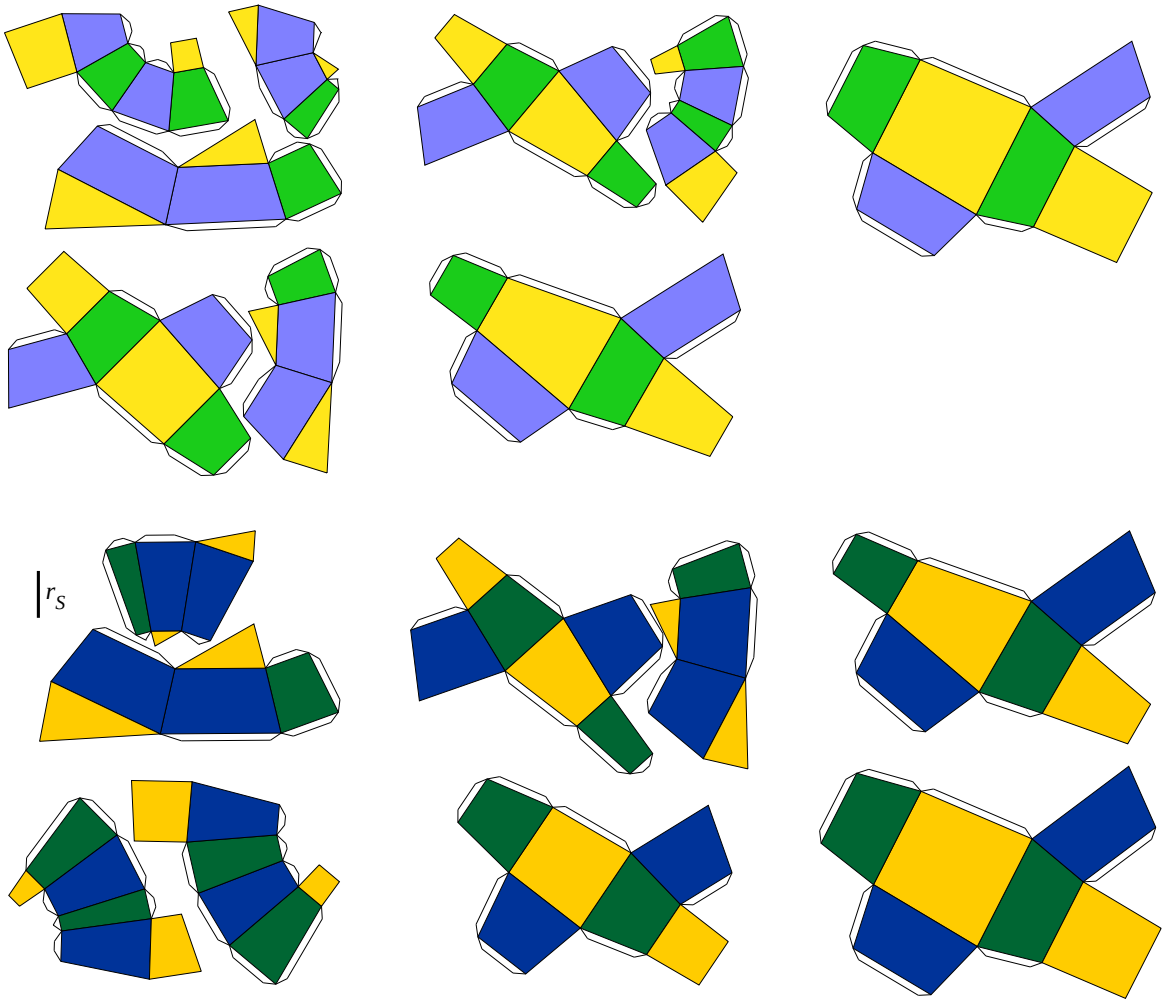
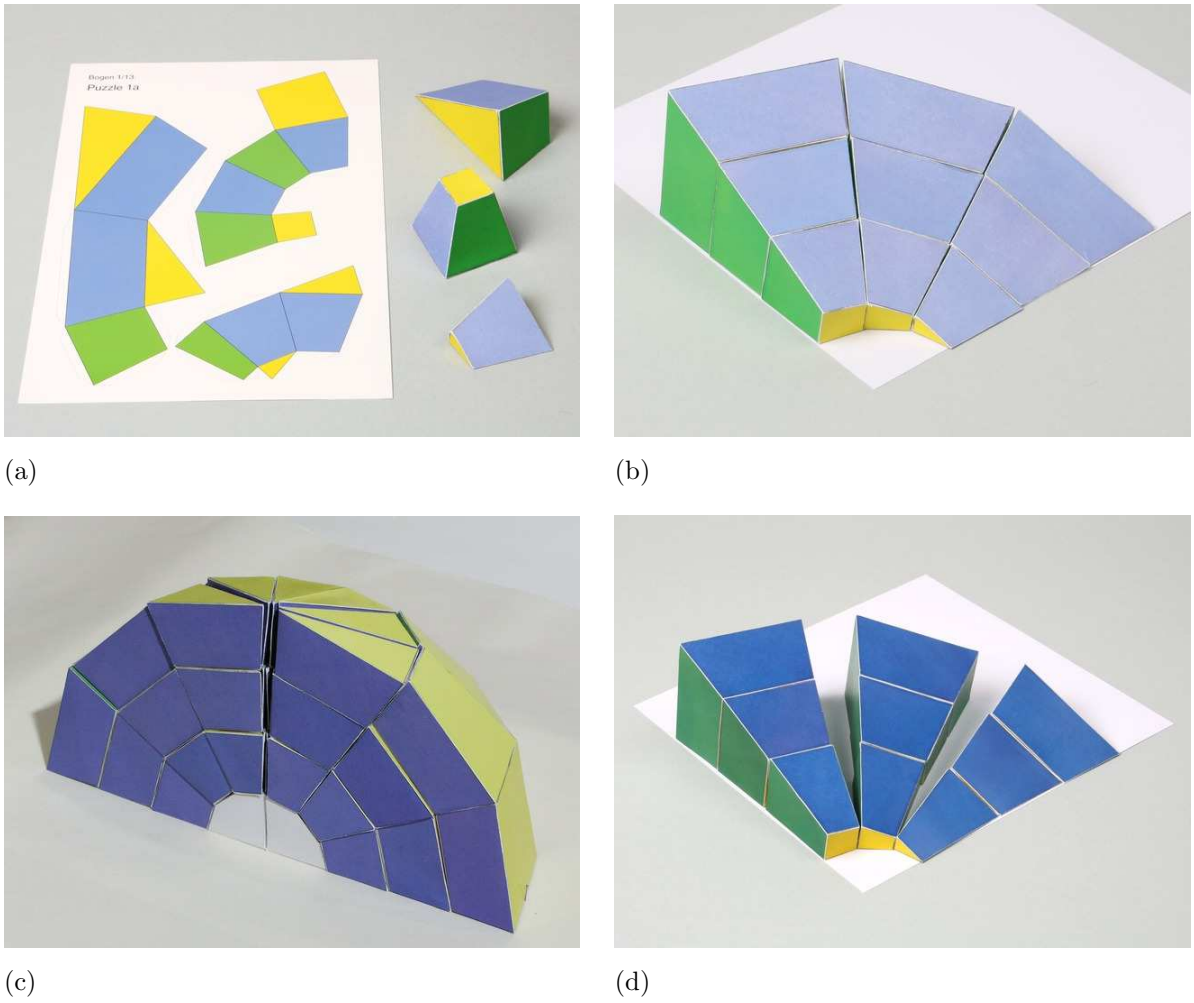


Figure 3. Cut-out sheets for cardboard models depicting euclidean space (top) and a black hole (bottom), respectively. The length of the scale bar indicates the Schwarzschild radius r_S of the black hole.

edges are measured, the blocks reproduced on a reduced scale and laid out in euclidean space. If the spatial region has zero curvature, the blocks can be assembled without gaps. Otherwise, the model indicates the curvature of the region.

Two such models – one depicting euclidean space, the other one depicting the curved space around a black hole – are built by the participants. The cut-out sheet for the euclidean model (figure 3 top) yields nine blocks with yellow, green, and blue sides (figure 4(a)). Assembled (yellow upon yellow and green upon green) this part of the model has the shape of half an orange slice (figure 4(b)). If possible, at least three such slices should be built. Set up side by side they form one eighth of a sphere (figure 4(c) shows one quarter of a sphere made up of six slices) with a small spherical cavity in the centre. In figure 4(c) an extra grey block fills the cavity to support the structure (a cut-out sheet for the supporting block is part of the online resources (Zahn and Kraus, 2014)). Twenty-four slices add up to the complete model of a hollow sphere.



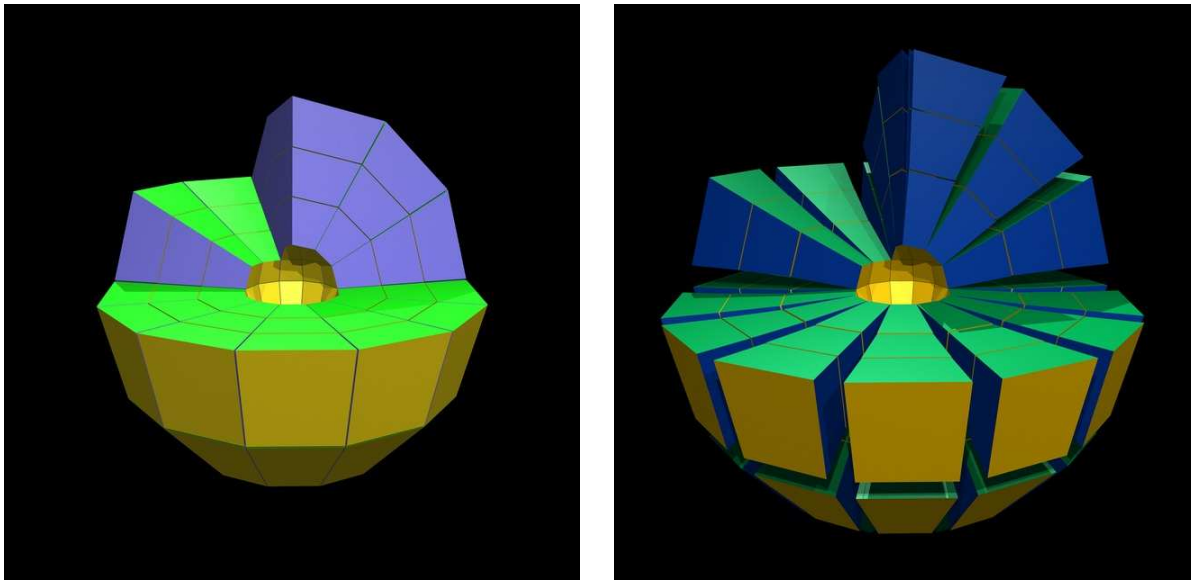
(a) (b)

(c) (d)

Figure 4. Sector models: individual sectors (a), single slice (b), and quarter sphere (c) of the model "euclidean space", single slice of the model "black hole" (d).

The computer generated picture in figure 5(a) shows the nearly complete sector model. Obviously all the blocks fit together perfectly as was to be expected for a euclidean space.

The second sector model depicts a spatial region of the same shape: A hollow sphere that has been subdivided into 24 slices of 9 blocks each, following the same pattern as before. This time, though, there is a black hole in the centre of the sphere so that space in this region is strongly curved. The reason for the spherical hollow at the centre of the models becomes clear at this point: The cavity is a little larger than the horizon of the black hole and so completely contains the interior region that cannot be represented by a rigid model. Using the cut-out sheet "black hole" (figure 3 bottom) nine blocks are built and combined to form a slice (figure 4(d)). Figure 5(b) shows a computer generated picture of the nearly complete model. The blocks obviously cannot be arranged to fill a sphere without gaps, and this reveals that the space in question has non-vanishing curvature.



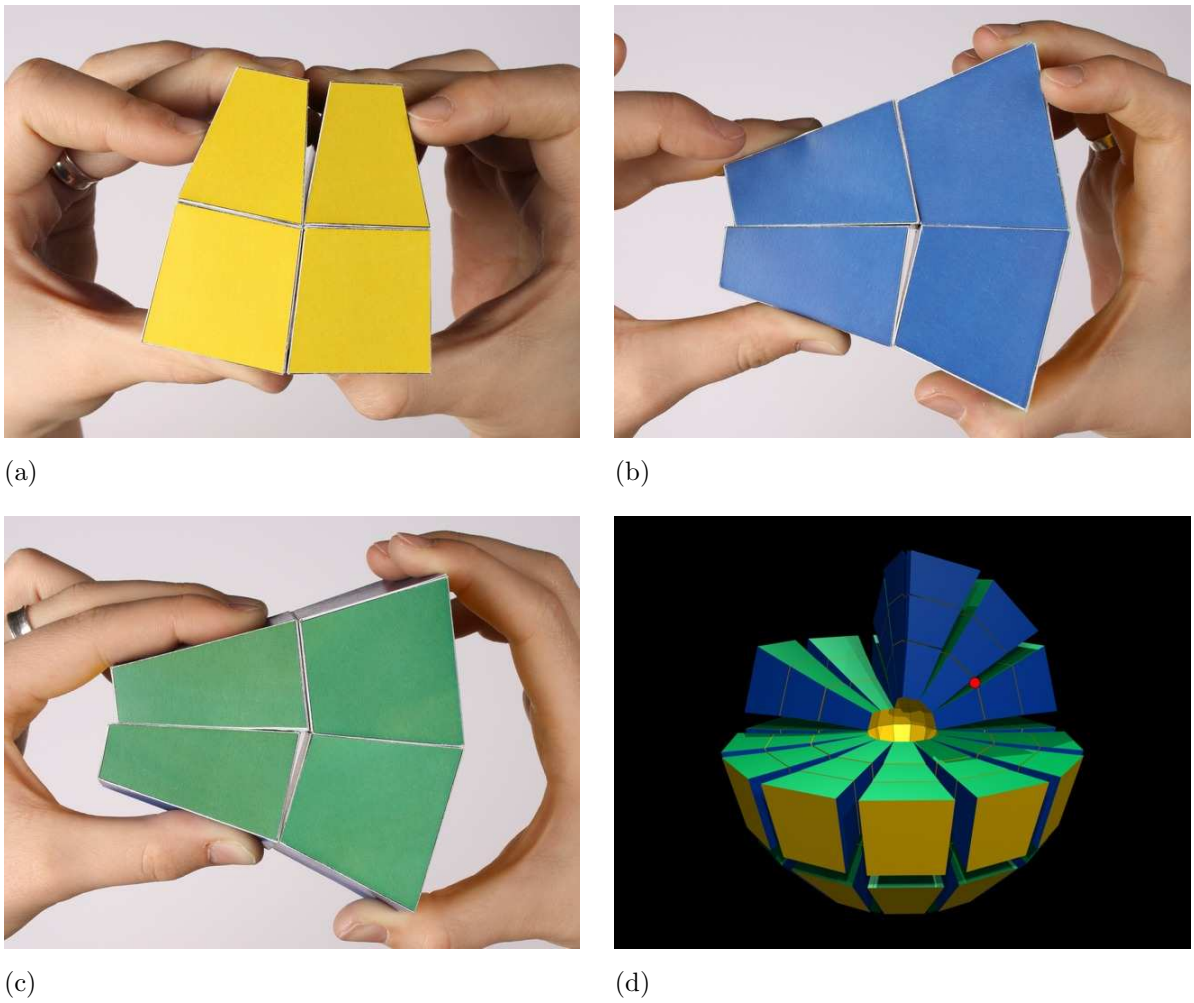
(a) (b)

Figure 5. Computer generated pictures of the sector models "euclidean space" (a) and "black hole" (b). If a black hole of the appropriate mass was placed in the centre of the black hole model, the blocks with the sizes and shapes as shown here would fit without gaps.

Gaps appear whenever the pieces of a curved surface are laid out in a plane or the blocks of a curved space are assembled in euclidean space. With a black hole of the appropriate mass in the centre of the black hole model, though, the blocks *the way they are* would fit without gaps. Just like the elements of area would fit without gaps if laid out on a surface with the appropriate curvature.

By means of the sector model, the curvature is then examined in more detail. This is done along the lines of the two-dimensional case: In figures 2(e) and 2(f), four elements of area each are joined at a common vertex in order to resolve the question of "tearing or buckling". In the spatial model, four blocks share not a vertex, but an edge. When all four are joined at the common edge, a gap may remain ("tearing", indicating positive curvature) as in figure 6(a) or the residual space after joining three blocks may be too small to hold the fourth ("buckling", indicating negative curvature) as in figures 6(b) and 6(c). Note that the three subimages (a), (b), and (c) of figure 6 belong to three different edges *at the same place*. I.e. the curvature at one and the same point has a different value and indeed a different sign depending on the orientation of the edge that is considered. Hence, in more than two dimensions, "curvature" is not a number, but a quantity with multiple components. It is one of the strong points of sector models that they clearly display this fundamental property of curved spaces of more than two dimensions (Zahn, 2008).

By comparing the two sector models one can moreover show that the laws of euclidean geometry do not hold in a curved space. For instance the relation between

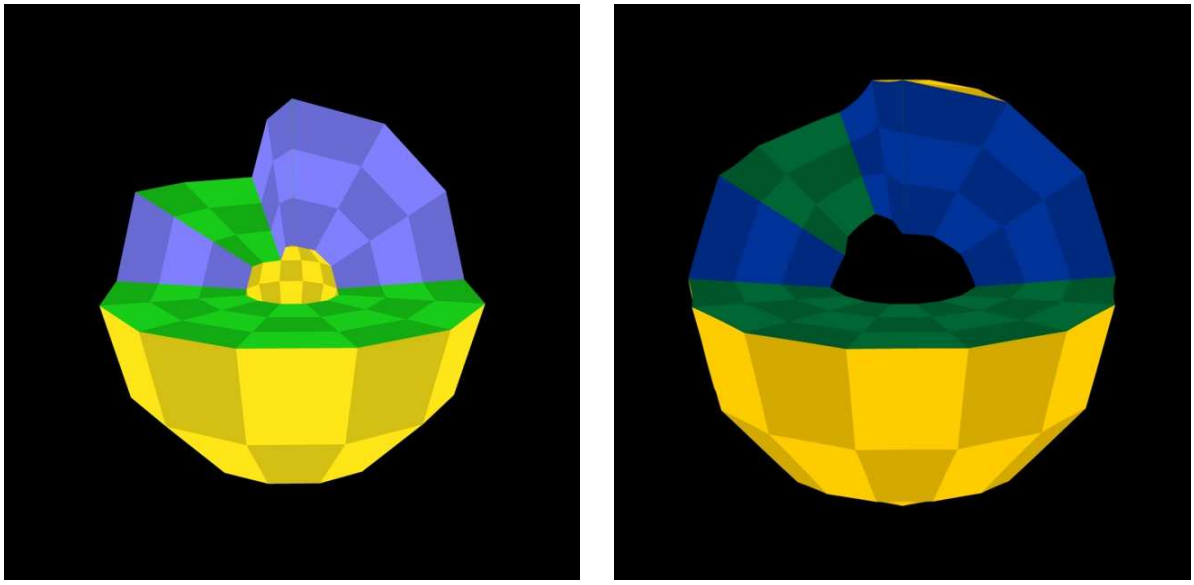


(a) (b)

(c) (d)

Figure 6. Curvature in three-dimensional space. The middle vertex in subimages (a), (b), and (c) is the same in all cases, it is marked in (d) with a red dot. The curvature is positive with respect to the radial edge (a) and negative with respect to both tangential edges (b, c).

surface and volume of an object: The outer spherical boundary of the models has the same surface area in both cases. This is easily seen by placing the outer surfaces of corresponding blocks side by side. The same holds for the surface area of the inner spherical boundary. The volume enclosed within these two boundaries is different, though: In radial direction, each block of the black hole model is longer than the corresponding block of the euclidean space model, i.e. the surface with the same area encloses a larger volume. At this point one may recall the flatlanders who in their two-dimensional world come to quite similar conclusions: When they measure circumference and area of a circle on a hill, the area is larger than it would be in the plane given the same circumference.



(a) (b)

Figure 7. View of the sector model "at its place of origin", (a) euclidean space, (b) Schwarzschild space.

2.3. Visualization of sector models

By construction, each sector model fills space completely, i.e. without gaps, at its place of origin. This also holds for the sector model of a black hole that in euclidean space cannot be assembled without gaps. One may ask what this sector model would look like, at its place of origin around a black hole, when the shape of each block is exactly as shown in figure 5(b) and these blocks join to fill a sphere. Figure 7(b) shows this view with the set-up of figure 5(b). The images in figure 7 have been computed with the ray tracing method of computer graphics by retracing, for each pixel, the incoming light ray to its point of origin (Zahn, 1991). As expected no gaps appear between the blocks‡. Comparing figure 7(b) to figure 5(b), one notices that the raytraced image appears somewhat distorted and in particular does not show the yellow inner spherical boundary. These effects are due to gravitational light deflection in the curved spacetime of the black hole.

3. The sector models

3.1. Sector models of curved surfaces

The surface with positive curvature presented in section 2.1 is a spherical cap with metric

$$\mathrm{d}s^2 = R^2\mathrm{d}\theta^2 + R^2\cos^2\theta\,\mathrm{d}\phi^2 \tag{1}$$

‡ In order to be able to distinguish neighbouring blocks, they have been given slightly different shades of colour in this image.

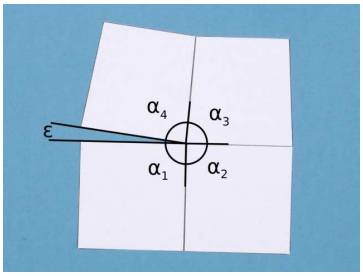


Figure 8. Deficit angle ϵ of a vertex.

and constant Gaussian curvature $K = 1/R^2$, R being the radius of the sphere. The surface with negative curvature is a saddle with metric

$$\mathrm{d}s^2 = R^2\mathrm{d}\theta^2 + R^2\cosh^2\theta\,\mathrm{d}\phi^2 \tag{2}$$

and constant Gaussian curvature $K = -1/R^2$. In order to subdivide the surfaces, (figures 2(c), 2(d)) vertices were chosen at coordinates

$$\begin{aligned} \theta_i &= -\frac{\pi}{6} + i\cdot\frac{\pi}{9} \qquad & i = 0\ldots 3, \\ \phi_j &= j\cdot\frac{\pi}{9} & j = 0\ldots 3. \end{aligned}$$

The lengths of the edges are the lengths of the spacelike geodesics between neighbouring vertices. They are computed by integrating the line element along the geodesics. The edge lengths together with the condition of mirror symmetry uniquely determine the shapes of the trapezoidal sectors.

The deficit angle of a vertex (figure 8) is defined as

$$\epsilon = 2\pi - \sum_i \alpha_i, \tag{3}$$

where the angles α_i are the interior angles of all the sectors containing the vertex in question. With increasing fineness of the subdivision into sectors

$$K = \rho\epsilon, \tag{4}$$

where ρ is the density of the vertices and ϵ their deficit angle, approximates the Gaussian curvature of the surface (Regge, 1961).

3.2. Spatial sector models

In section 2.2 sector models of euclidean space and of the space around a black hole were introduced. The respective metrics are

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{5}$$

for euclidean space and

$$\mathrm{d}s^2 = \left(1 - \frac{r_\mathrm{S}}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{6}$$

for the three-dimensional spacelike hypersurface of the Schwarzschild spacetime that is defined by constant Schwarzschild time. Here, $r_\mathrm{S} = 2GM/c^2$ is the Schwarzschild radius

of mass M, G the gravitational constant, and c the speed of light in vacuum. The sector model built from the cut-out sheets available in the online resources (Zahn and Kraus, 2014) corresponds to a black hole mass of three earth masses if the sheets are printed in A4 format.

For the partitioning of the spherical regions (figure 4) vertices were chosen with coordinates

$$\begin{aligned} r_i &= i \cdot 1.25\, r_\mathrm{S} && i = 1 \ldots 4, \\ \theta_j &= j \cdot \frac{\pi}{6} && j = 0 \ldots 6, \\ \phi_k &= k \cdot \frac{\pi}{6} && k = 0 \ldots 11. \end{aligned}$$

Partitioning is limited to the region outside $r = 1.25\, r_\mathrm{S}$. Inside r_S no static spacelike hypersurface can be defined. The computation is performed as in the case of the curved surfaces: The lengths of the edges between neighbouring vertices are computed by integrating the line element along spacelike geodesics. In accord with the symmetric partitioning of the spherical region, the sector faces are described as isosceles trapezia. The shapes of the faces are uniquely defined by the edge lengths together with this symmetry condition. The complete spatial sector model is built up of 216 sectors. Since the partitioning consists of 24 identical slices, one slice contains the full information on the geometry. The cut-out sheets comprise the nine sectors of one slice, corresponding to $i = 1 \ldots 4$, $j = 0 \ldots 3$ and $k = 0 \ldots 1$. The sector models of euclidean space and Schwarzschild space are true-to-scale representations of the respective metrics. The models therefore display the geometric properties of these spaces in a way that is quantitatively correct (within the framework of the rather coarse discretization). In particular in the radial direction a sector of the Schwarzschild space model is longer than the corresponding sector of the euclidean space model. They differ by a factor given by the integral of the metric coefficient $\int (1 - r_\mathrm{S}/r)^{-1/2}\, \mathrm{d}r$ along the edge.

The deficit angle with respect to some edge is again defined by equation (3). In euclidean space it is zero, in Schwarzschild space it depends on the orientation of the edge (figure 6). In a local orthonormal coordinate system and for a partitioning into coordinate cubes, the components $R^{\hat{j}}{}_{\hat{k}\hat{j}\hat{k}}$ of the Riemann curvature tensor are approximated as

$$R^{\hat{j}}{}_{\hat{k}\hat{j}\hat{k}} = \rho \epsilon l, \tag{7}$$

where $\boldsymbol{e}_{\hat{i}}$, $\boldsymbol{e}_{\hat{j}}$, $\boldsymbol{e}_{\hat{k}}$ are the basis vectors, ρ is the density of the edges in i-direction, ϵ their deficit angle and l their length (Regge, 1961; Misner, Thorne and Wheeler, 1973).

4. Spacetime sector models

The idea of approximating a curved surface or a curved space by small uncurved sectors can also be applied to curved spacetimes. In this case the uncurved sectors have Minkowski geometry. Figure 9 gives an example of a spacetime sector model. The curved spacetime here is the two-dimensional t-r subspace of the Schwarzschild spacetime with

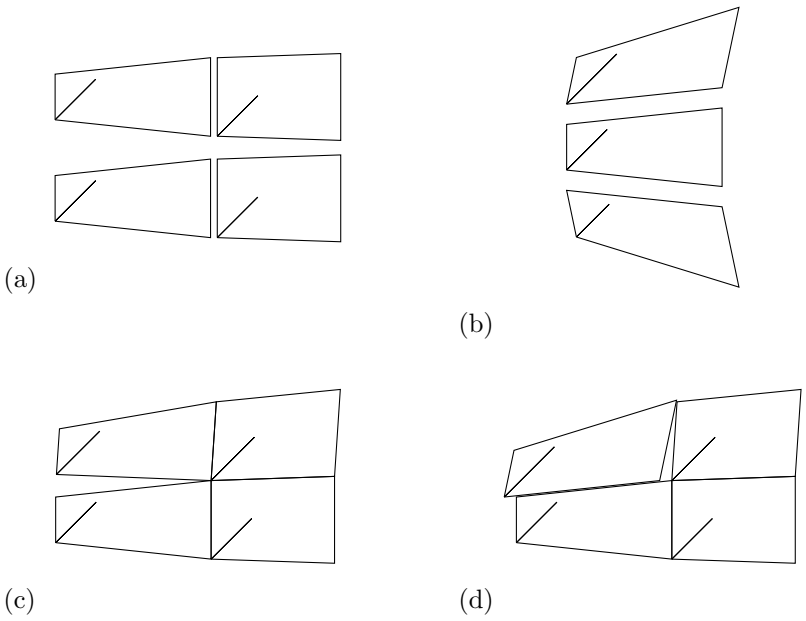


Figure 9. (a) Spacetime sector model of the t-r subspace of the Schwarzschild spacetime. The spacelike coordinate axis is horizontal, the timelike one vertical, and the diagonal lines mark the light cones. (b) Different representations of the same spacetime sector. Lorentz transformations convert them into each other. (c) Deficit angle in spacelike direction: "tearing". (d) Deficit angle in timelike direction: "buckling".

the Schwarzschild radial coordinate r as space coordinate and the Schwarzschild time t as time coordinate. The metric of this subspace is

$$\mathrm{d}s^2 = -\left(1 - \frac{r_\mathrm{S}}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{r_\mathrm{S}}{r}\right)^{-1}\mathrm{d}r^2 \tag{8}$$

with the Schwarzschild radius r_S. In this section geometric units are used so that the speed of light in vacuum c is equal to unity.

The vertices for the subdivision into sectors are at

$$\begin{aligned} t_i &= i \cdot 1.25\, r_\mathrm{S} \qquad i = 0 \ldots 2, \\ r_j &= j \cdot 1.25\, r_\mathrm{S} \qquad j = 1 \ldots 3. \end{aligned}$$

The geodesics that connect neighbouring vertices are partly spacelike and partly timelike. In both cases the length of the edges is determined by integrating the line element along the geodesics. The metric coefficients do not depend on the t coordinate, therefore identical sectors are lined up in t direction. Each sector is described as an isosceles trapezium with the timelike edges as bases. This symmetry condition together with the edge lengths uniquely determines the shapes of the sectors. Fig. 9(a) shows the representation of the sectors in a Minkowski diagram.

The geometry in the interior of each sector is Minkowskian and the statements of the special theory of relativity about Minkowski spacetime apply. In particular this includes the free choice of the inertial frame in which events are described and the use of

the Lorentz transformation in order to change from one inertial frame to another. When two inertial frames connected by a Lorentz transformation are visualized in a Minkowski diagram, their time axes form an angle, the space axes likewise but in the opposite direction, and the light cone is common to both frames. Thus, the representation of the sectors in a Minkowski diagram as shown in figure 9(a) corresponds to the choice of a certain inertial frame, and a boost transformation into a different inertial frame changes the representation of the sectors (figure 9(b)). In the spatiotemporal sense, though, the sectors have the same shape and symmetry in all inertial systems, since these are described in terms of scalar products that are invariant with respect to Lorentz transformations. In order to illustrate the curvature in this subspace, sectors must again be assembled around a vertex. In a spatial sector model a sector is rotated in order to lay it alongside the neighbouring sector. In the spatiotemporal case the rotation is replaced by a Lorentz transformation. One can see that joining neighbouring sectors by means of rotation in the t-r plane is not possible by considering the light cones that must be identical in the two sectors that are joined. Since a change of inertial frame entails a change in the representation of a sector as described above, the common edge of neighbouring sectors can be made to coincide in the Minkowski diagram by a Lorentz transformation (figure 9(c)).

In case of a global Minkowski spacetime, all the sectors can be combined in this way to cover without gaps a section of the spacetime. In the t-r subspace of the Schwarzschild spacetime, however, there is a deficit angle at the vertex (figure 9(c)), indicating there is a non-zero spatiotemporal curvature. If a gap appears between two spacelike edges (as in this example), the spatiotemporal curvature is positive, in case of an overlap it would be negative§. If the sectors are arranged in such a way that the deficit angle appears between two timelike edges, an overlap results in the example shown here (figure 9(d)). The gap between the spacelike edges and the overlap of the timelike edges specify the same Lorentz transformation that would make the edges on both sides of the deficit angle meet.

Similar to the spatial case, the deficit angle is related to the respective component of the curvature tensor: In a local orthonormal coordinate system, and for a partitioning into coordinate cubes, the component $R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}}$ is approximated as

$$R^{\hat{t}}{}_{\hat{r}\hat{t}\hat{r}} = \rho\alpha, \tag{9}$$

where $\boldsymbol{e}_{\hat{t}}$ and $\boldsymbol{e}_{\hat{r}}$ are the timelike and spacelike basis vectors, respectively, ρ is the local density of vertices and α the rapidity of the Lorentz transformation specified by the local deficit angle (the velocity being $\beta = \tanh\alpha$).

§ Note that when the signature is chosen to be $(+\,-\,-\,-)$, the spatiotemporal curvature has the opposite sign: It is negative (positive), if a gap (an overlap) appears between two spacelike edges.

5. Conclusions and outlook

Sector models are physical models that represent two- and three-dimensional curved spaces or spacetimes true to scale. This description of curved spaces does not use coordinates, it therefore provides physical insight in a direct and intuitive way. We have shown how the curvature of a three-dimensional space and of a 1+1-dimensional spacetime can be described by means of such models. This is the answer that we propose to give to the first of the three questions raised in the introduction, "What is a curved spacetime?". The work with the sector models can stand on its own, or it can supplement the usual mathematical introduction of the Riemann curvature tensor by providing a visualization.

The workshop on curved spaces outlined in section 2 has been held a number of times with undergraduate students, high school students and interested adults (Zahn and Kraus, 2004; Kraus and Zahn, 2005; Zahn and Kraus, 2010, 2012). The programme and the materials have been developed in several cycles of testing and revision. For several years, this workshop has been held at Hildesheim university for students who intend to become physics teachers.

In comparison with the introduction of curvature via prototypical curved surfaces as described in the introduction, the use of sector models has a considerably wider scope. Curved two-dimensional surfaces when depicted in the usual way are shown embedded in three-dimensional space. This picture cannot be carried over to curved spaces because the necessary higher dimensional embedding space cannot be visualized. The sector models, in contrast, need no extra dimension. The two-dimensional case (flatland) is carried over directly to the three-dimensional case and so makes the curved three-dimensional case accessible to "spacelanders". In particular, sector models illustrate the tensorial character of curvature in spaces of more than two dimensions. This can be accomplished neither with prototypical curved surfaces nor with embedding diagrams.

The usefulness of sector models extends well beyond the visualization of curvature as will be shown in a sequel to this paper. Sector models permit to introduce other geometric concepts (e.g. parallel transport, geodesics) in an intuitive way. They can be used to study relativistic phenomena by construction instead of computation (e.g. particle paths, redshift). Other spacetimes that e.g. contain matter or are not static can be described in the same way as the Schwarzschild spacetime considered here. Thus, the second and third of the fundamental questions stated in the introduction, namely the motion of particles and the connection between curvature and matter can also be addressed within this framework.

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