Teaching gravitational light deflection: A short path from the metric to the geodesic

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Gravitational light deflection was one of the early tests of the general theory of relativity and has by now attained a huge importance in astronomy. When teaching this topic in school, however, one is faced with the problem that the theoretical description of light paths as geodesics relies on advanced mathematics. In this contribution we describe a method that allows students to determine the geodesics for a given metric by means of a graphic construction.

Introduction

Matter tells space how to curve.
Space tells matter how to move.
(John Wheeler)

By the end of the year 1915 Albert Einstein completed the general theory of relativity that describes gravity in terms of the geometry of spacetime. The distribution of matter determines the geometry of the spacetime, and the geometry determines the motion of matter: Gravitation is geometry!
Barely two months later, Karl Schwarzschild found a solution for the geometry of spacetime outside of a spherically symmetric mass [1]. This solution of Einstein's field equations describes the gravitational effects of spherical celestial bodies like planets, stars, neutron stars, and black holes.

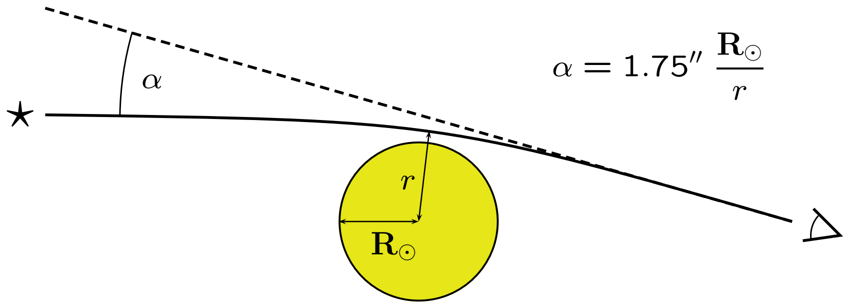


Abbildung 1: Light deflection close to the sun. A distant star (left) is seen displaced by an angle α. In this diagram the angle is drawn out of scale. 1.75 seconds of arc correspond to a hair's breadth seen from a distance of 10 metres. The angle of deflection decreases with increasing distance r.

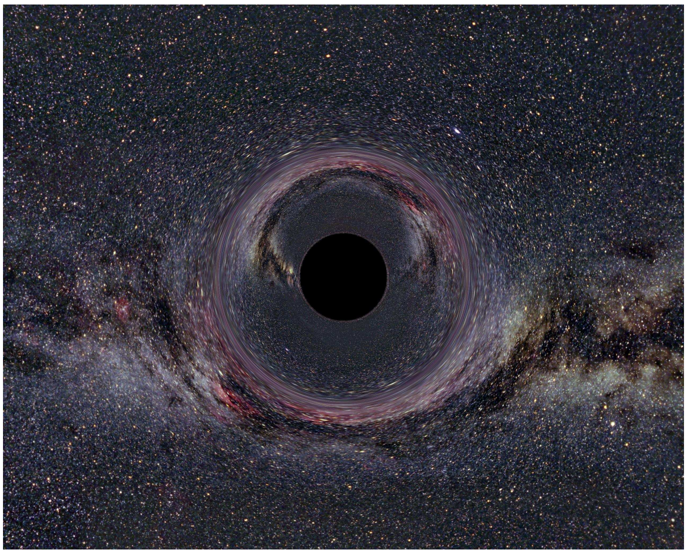


Abbildung 2: In the computer simulation a black hole is located in front of the luminous band of the Milky Way. From the interior region of the black hole, the region inside the so-called event horizon, neither light nor matter can escape; this appears as a black disk.

The paths of particles and light are determined by the geometry of spacetime. In a curved spacetime, particles and light follow locally straight lines, the so-called geodesics. One of the many implications of Schwarzschild's solution is the deflection of light: When light passes close by a spherically symmetric mass, then its direction after the encounter is different from the direction before it. This prediction was of historical importance: The confirmation of light deflection close to the sun (Fig. 1) by Arthur Eddington in 1919 was an important test of the then new gravitational theory. For a light ray grazing the sun, the deflection amounts to 1.75 seconds of arc. This is a very small angle and its measurement was then difficult. The reason for its smallness is the very weak curvature of spacetime in our astronomical neighbourhood; the geometry is nearly Euclidean. Deflection by large angles occurs for example close to a black hole. There, the optical distortions are so large that they would be visible to the naked eye (Fig. 2).

This contribution presents an approach to the effect of gravitational light deflection that is suitable for use in schools. It is on the one hand close to the theory because it is based on the description of spacetime by a metric and on the determination of the paths of particles and light as geodesics, i. e. as locally straight curves. On the other hand the general theory of relativity is a geometric theory and can therefore be understood in geometric terms. The materials that we present stress the geometric aspect and in this way make do with elementary mathematics. Using a black hole as an example, we explain how curved space is described by means of a metric in general relativity. We describe a graphic method that enables students to determine geodesics for a given metric on their own.

The metric as a tool for the description of surfaces

In this section we describe an introduction to the concept of a metric. The starting point is the computation of the distance between neighbouring points. This is considered first in the plane with Cartesian coordinates, then in the plane with polar coordinates, and

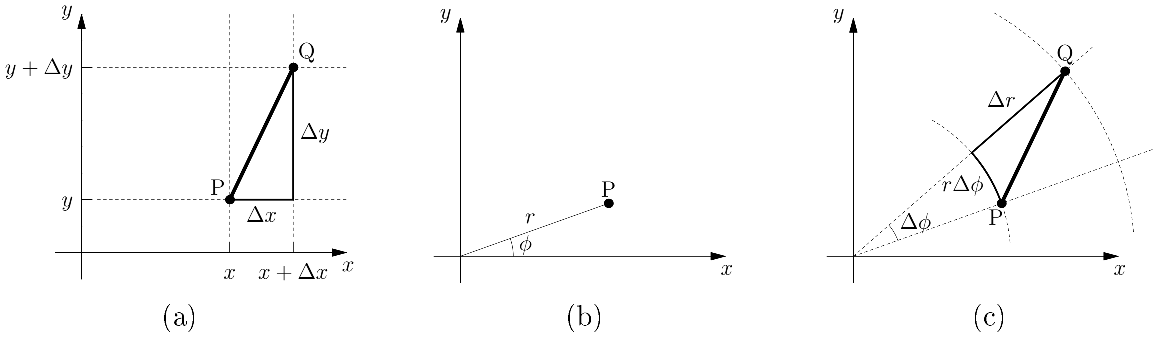


Abbildung 3: Examples of different metrics. a: Distance in Cartesian coordinates, b: Definition of polar coordinates, c: Distance in polar coordinates.

finally on the surface of the sphere. Following these three examples of metrics we study the inverse problem: Given a metric, what does it tell us about the surface?

In a plane surface, point P is assigned Cartesian coordinates x and y. A neighbouring point Q has coordinates that are only slightly different: $x + \Delta x$ and $y + \Delta y$. We raise the question of the distance between the two points. From the rectangular triangle in Fig. 3a one finds that the distance Δs satisfies

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2. \tag{1}$$

When the differences in the coordinates are given, then equation (1) provides the distance. A function of this type is called a metric. Equation (1) specifies the metric of a plane surface in Cartesian coordinates.

Polar coordinates are another possibility of labelling the points of a plane surface. A point P is assigned a radial coordinate r and an azimuthal angle ϕ (Fig. 3b). Again, we raise the question of the distance to a neighbouring point Q. If Q has the same azimuthal angle but a different radial coordinate $r + \Delta r$, then the distance between the points is Δr. If Q has the same radial coordinate, but a different azimuthal angle $\phi + \Delta\phi$, then the circular arc between the two points has length $r\Delta\phi$. For points that are very close together, the small section of circular arc between them is approximately straight and is practically equal to the distance between the points. If both coordinates differ, then with the help of the rectangluar triangle in Fig. 3c one finds that

$$(\Delta s)^2 = (\Delta r)^2 + r^2(\Delta\phi)^2. \tag{2}$$

Equation (2) specifies the metric of a plane surface in polar coordinates.

Distances on curved surfaces are described in the same way. For the points on the surface of a sphere with radius R for instance, designated by the usual spherical coordinates θ and ϕ, the distance of neighbouring points satisfies

$$(\Delta s)^2 = R^2(\Delta\theta)^2 + R^2\sin^2\theta\,(\Delta\phi)^2. \tag{3}$$

Equation (3) specifies the metric of the surface of a sphere in spherical coordinates. These three examples illustrate what a metric can do: It permits to compute the distance

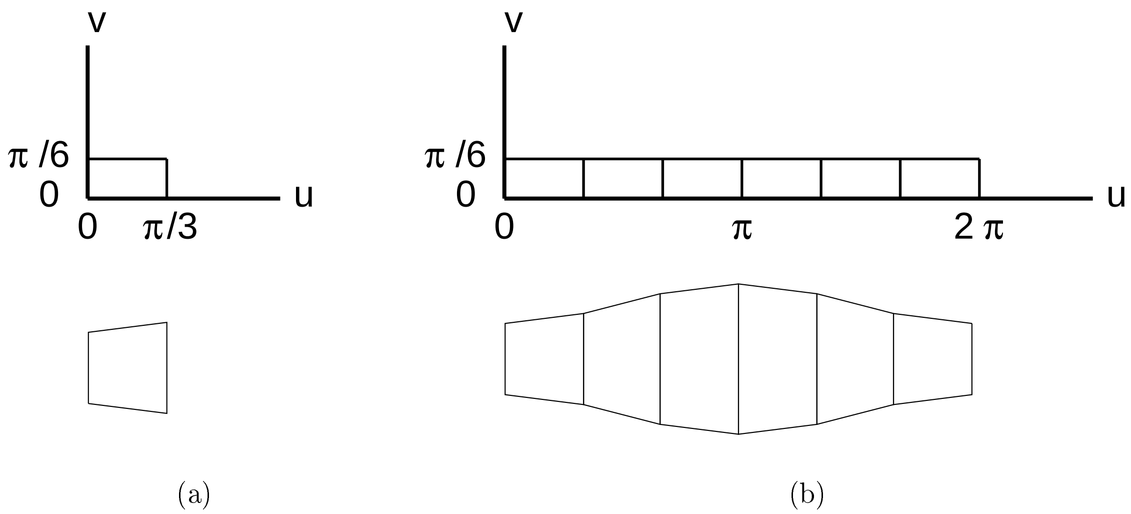


Abbildung 4: Investigation of a metric. a: a rectangle in coordinate space and the corresponding quadrangle in the surface, b: a strip in coordinate space covering the complete u-range and the corresponding strip in the surface

of neighbouring points when the differences in their coordinates are known. At first sight, this may appear to be no big deal. But in fact the metric contains the complete information on the interior geometry of the surface.

The following exercise can be used to show quite strikingly how much the metric reveals about a surface. We ask the following question: The only thing known about a surface is its metric given by

$$(\Delta s)^2 = b^2(\Delta u)^2 + (a - b\cos u)^2(\Delta v)^2. \tag{4}$$

Here u and v are coordinates on the surface, each with values in the range from 0 to 2π. The quantities a and b are constants, in this example their values are $a = 7$ cm und $b = 2.5$ cm. What is this surface like? What is the geometrical significance of the coordinates?

We start with a small section of the surface. In u it covers one sixth of the coordinate range and in v one twelfth (Fig. 4a top). The rectangle in coordinate space corresponds to a quadrangle in the surface, the latter, however, need not be rectangular. Size and shape of the quadrangle in the surface are found by computing the lengths of the four edges; this is possible by use of the metric. The upper and lower edges are on lines of constant v and both have the length

$$\Delta s = b\,\Delta u = b\,\pi/3 \qquad (\Delta v = 0). \tag{5}$$

The lateral edges have constant u and length

$$\Delta s = (a - b\cos u)\,\Delta v = (a - b\cos u)\,\pi/6 \qquad (\Delta u = 0). \tag{6}$$

For the left edge with $u = 0$ this results in $\Delta s = (a - b)\,\pi/6$, for the right edge with $u = \pi/3$ one finds $\Delta s = (a - b/2)\,\pi/6$. The upper and lower edge being of the same

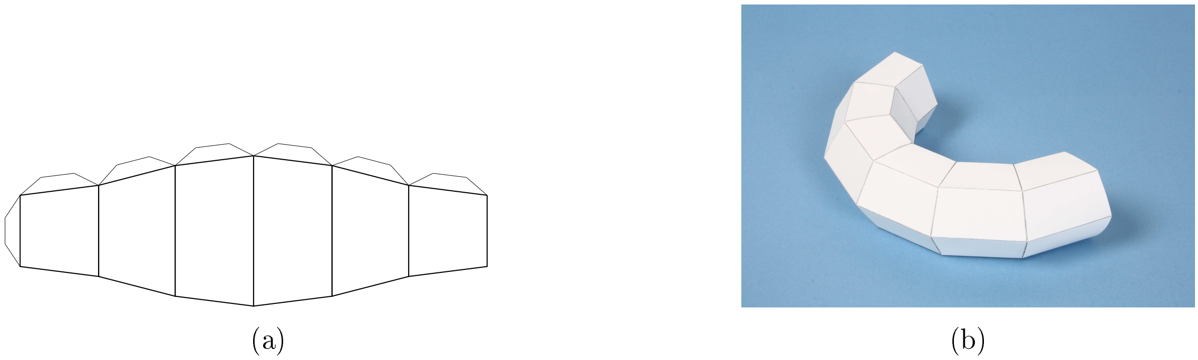


Abbildung 5: Torus. a: Strip with glue laps for building the surface, b: The surface built up of facets as constructed from the metric.

length, the quadrangle is constructed as a symmetric trapezium (Fig. 4a bottom): This is the small sector of the surface that corresponds to the chosen coordinate patch. In the second step this computation is extended onto the whole u-range. For five additional coordinate patches, each covering one sixth of the u-range (Fig. 4b top) the corresponding sectors of the surface are determined. The result is the strip shown in Fig. 4b (bottom). Finally, the whole v-range is to be covered. It is subdivided into twelve segments of equal coordinate length. Each segment corresponds to a strip of six sectors on the surface. Since the metric does not depend on the coordinate v, the twelve strips are all identical.

By means of the glue laps added in Fig. 5a the surface can be assembled from twelve strips. A cut-out sheet is available online [2] as supplement to this contribution. Assembling a few strips is enough to recognize the surface: Glued together the strips bend into doughnut shape. When each strip is closed into a ring (by identifying $u = 0$ and $u = 2\pi$) and the twelfth strip is joined to the first (thus identifying $v = 0$ and $v = 2\pi$), the result is a torus. The assembled surface also discloses the geometric meaning of the coordinates: A cross section of the tube is a circle with radius b and angular coordinate u. The centerline of the tube is a circle with radius a and angular coordinate v.

This example shows which properties of the surface come out of the metric and which don't. The way the surface bends follows perforce from the dimensions of the sectors, and these are completely determined by the metric. The geometric meaning of the coordinates becomes apparent on the reconstructed surface. Global conditions, however, like the fact that a surface is closed, must be specified in addition to the metric.

In the construction of the sectors, the vertices are treated as neighbouring points the distance of which is specified by the metric. Accordingly, the sectors themselves are treated as small and being approximately plane. With these approximations the surface is built up of small plane facets. The finer the subdivision into patches the better the surface is approximated.

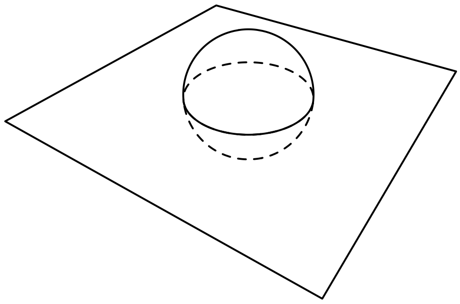


Abbildung 6: A symmetry plane of the space around a black hole.

Schwarzschild metric: The equatorial plane of a black hole

In this section the metric of the space around a black hole is investigated with the same method as used above for the metric of the torus. The result is a physical model that is subsequently used to construct geodesics. Due to the spherical symmetry of the black hole each geodesic is confined to a plane; these planes are symmetry planes of the space around the black hole (Fig. 6). An investigation of geodesics may therefore be restricted to such a symmetry plane, in the following called equatorial plane. The metric of this plane is

$$(\Delta s)^2 = \frac{1}{1 - r_{\rm S}/r}(\Delta r)^2 + r^2(\Delta\phi)^2, \tag{7}$$

with the so-called Schwarzschild radius $r_{\rm S}$ of the black hole that quantifies its mass: $r_{\rm S} = 2GM/c^2$ with Newton's gravitational constant G and the speed of light c. In case of zero mass, this is identical with equation (2) for the metric of the plane in polar coordinates. The coordinate ϕ is the azimuthal angle for which the values zero and 2π are identified, r is the radial coordinate.

Using the metric one can "build" the equatorial plane in the same way as described above for the torus. For the model presented below, the ϕ-range is subdivided into twelve segments of coordinate length $\pi/6$ each. Since the metric does not depend on coordinate ϕ, the sectors need only be computed for one of the segments.

In r, we consider the range between $1.25\,r_{\rm S}$ and $5\,r_{\rm S}$, just outside the event horizon ($r = r_{\rm S}$). It is subdivided into three segments of coordinate length $1.25\,r_{\rm S}$ each (Fig. 7a left). One needs to compute the lengths of the edges of the three quadrangles. The distance between vertices with the same r coordinate is computed as above as the length of the circular arc:

$$\Delta s = r\,\Delta\phi \qquad (\Delta r = 0). \tag{8}$$

In between vertices with the same ϕ coordinate, the metric coefficient $1/(1 - r_{\rm S}/r)$ comes into play. This coefficient depends on r, therefore it varies along the edge. For a simple computation that gives a good approximation to the edge length, this coefficient is evaluated at the mean r-coordinate $r_{\rm mi}$ of the edge:

$$\Delta s = \sqrt{\frac{1}{(1 - r_{\rm S}/r_{\rm mi})}}\,\Delta r \qquad (\Delta\phi = 0). \tag{9}$$

The two edges on lines of constant ϕ have the same length. This reflects the fact that the metric is independent of ϕ. Because of this symmetry, the sectors are constructed as

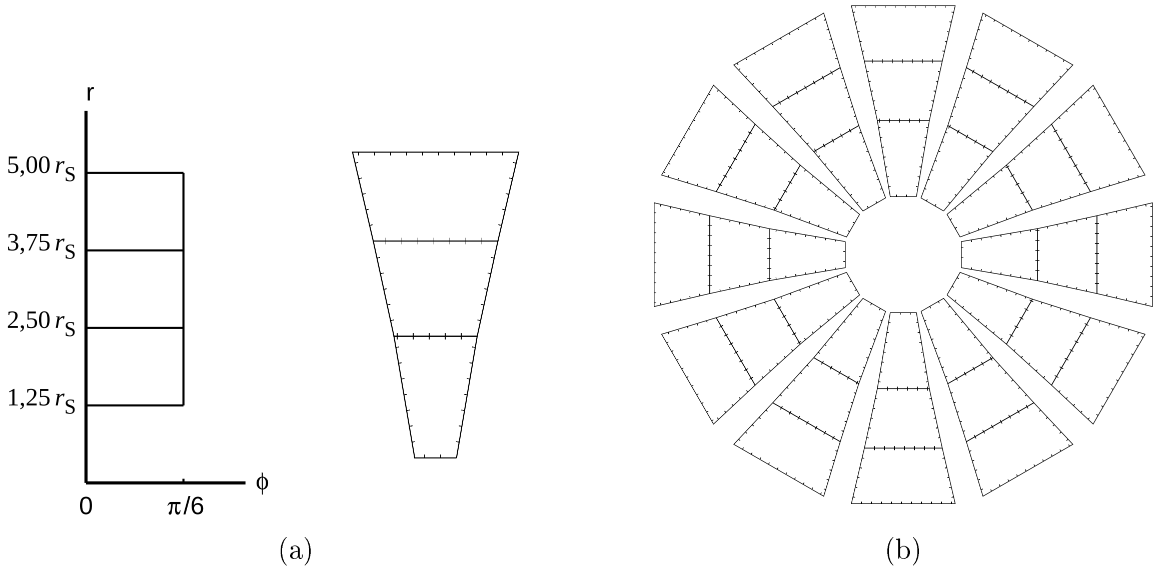
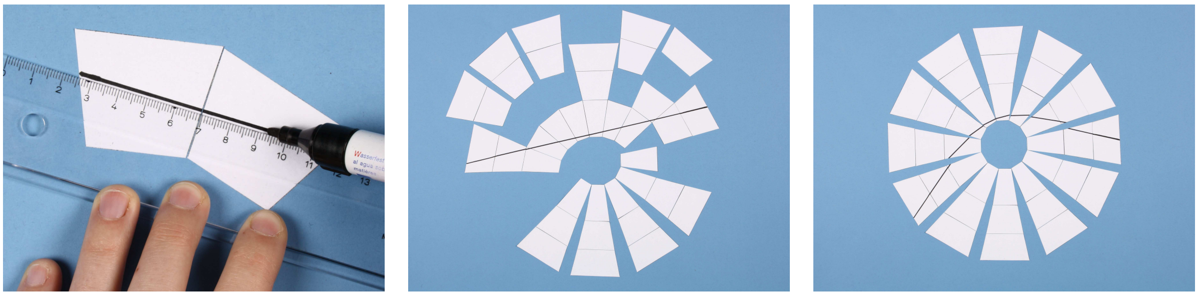


Abbildung 7: Geometry of the equatorial plane of a black hole. a: a column made up of three rectangles in coordinate space and the corresponding column of three quadrangles in the surface, b: the surface is represented by 12 identical columns.

symmetric trapezia. Fig. 7a (right) shows the result for one column, Fig. 7b shows the complete model made up of twelve identical columns that cover the whole range of the azimuthal angle ϕ. The sectors form three concentric rings. One notices that the sectors cannot all be joined without leaving gaps. This signals that they describe a plane that is part of a curved space. In Fig. 7b the sectors are displayed on a plane surface that is part of a Euclidean space. If one could place a black hole of the appropriate mass into the center of the model, the sectors *the way they are* would fit together without gaps. The model that is available online at [2] has 27 cm diameter when used in an A3 format, the appropriate black hole has 2.4 earth masses.

We call a model of this kind, representing a curved space by the use of small uncurved pieces, a *sector model*. With sector models, different aspects of general relativity can be illustrated in a non-mathematical way [3, 4].

As a note to Fig. 7b we should like to mention that the sectors can be glued together in the case of the equatorial plane as well. The result is a surface in the shape of a funnel. The interior geometry of this surface is indentical with the interior geometry of the equatorial plane. This is the so-called embedding diagram of the equatorial plane, also known als Flamm's paraboloid. Thus, the construction of sector models can also be used to obtain embedding diagrams. It is our experience, however, that the concept of the embedding diagram is difficult to teach. The representation is often misunderstood as the geometric shape of the object ("a black hole is a funnel"). We therefore do not introduce embedding diagrams in our workshops, instead we work exclusively with the plane representation shown in Fig. 7b.



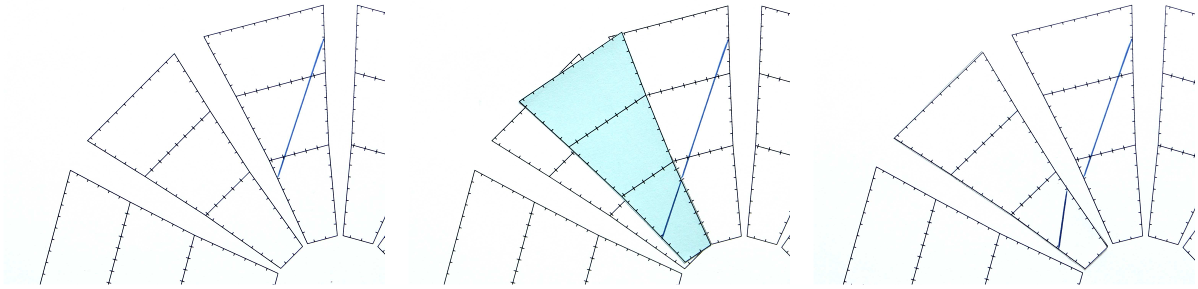
(a) (b) (c)

Abbildung 8: Geodesics in the equatorial plane of a black hole. a: Geodesics are locally straight, b: construction of a geodesic, c: geodesic on the symmetrically displayed model.

Geodesics in the equatorial plane of a black hole

According to the description given by the general theory of relativity, light propagates locally in a straight line. I. e. at each point on the path, the current direction is maintained, there are no bends and no kinks. The same holds for the motion of free particles. A locally straight line is called a geodesic. Therefore, in order to find the paths of particles or light, one must determine geodesics. On a sector model, this is simple: A sector being a chunk of uncurved space, in the interior of a sector a geodesic is a straight line; when the line reaches the border of a sector it is continued onto the neighbouring sector. How to continue it is specified by the definition: locally straight (Fig. 8a). Using this drawing rule a geodesic can be constructed across the equatorial plane (Fig. 8b). On the symmetrically displayed model (Fig. 8c) one can see that the direction "far behind" the black hole differs from the direction "far ahead" of the encounter. The construction shows that a line that is locally straight at each point and that passes through a region of curved space enters this region in some direction and leaves it in a different direction. In order to assess the significance of this construction two things must be borne in mind. Firstly, the geodesic constructed as shown above is correct in the sense that it is a solution of the geodesic equation. Since the sector model is an approximate description of the curved space, this geodesic is an approximate solution. A geodesic can in principle be constructed with high accuracy by using a fine subdivision into appropriately small sectors. Secondly, though the line constructed above is a geodesic, it is not a light ray. This geodesic is a line in space. But light propagates through space and time, i. e. light paths are geodesics in spacetime. Nevertheless, the geodesic in space illustrates the principle behind gravitational light deflection.

By use of the sector model one can study the properties of geodesics (in space) in more detail. One can, for instance, show that geodesics are deflected more strongly if they pass closer by the black hole. One can construct two geodesics that initially are close together and parallel and one finds that the distance between them increases; this indicates that the equatorial plane has negative curvature. Also, one can ascertain that it is possible to draw geodesics that form a digon: Two geodesics starting at the same point that pass the black hole on opposite sides and intersect on the far side. In case of light rays, double



(a) (b) (c)

Abbildung 9: The construction of geodesics using transfer sectors. a: The geodesic is drawn up to the border of the column, b: ... is continued onto the transfer column c: ... and is copied from the transfer column onto the neighbouring column.

images are produced in this way.
It is possible to perform these constructions as shown in Fig. 8b: The sectors are cut out of a sheet of paper, are glued onto cardboard with spray adhesive and are arranged along a geodesic or in symmetric display as required.
A second method is simpler and faster: The model is used in symmetric layout as shown in Fig. 7b, best enlarged to an A3 format (online available [2] as supplementary material to this contribution). In addition one needs a single column that is cut out; these are the so-called transfer sectors. Starting on the symmetrically displayed model the geodesic is drawn up to the border of the column (Fig. 9a). At this point the appropriate transfer sector is appended and the line is continued straight across the column of transfer sectors (Fig. 9b). The line is then copied from the transfer sectors onto the neighbouring column of the symmetric model (Fig. 9c). This procedure is continued until the desired end point is reached.
The graphic construction of geodesics can be extended to spacetimes. It is then possible to construct world lines of photons and free particles and so for instance to study gravitational redshift. There are other aspects of general relativity that can be described with the use of sector models. In particular one can illustrate curvature not only of surfaces but also of three-dimensional curved spaces and of spacetimes. More about sector models and more material can be found on [5]; this collection of models and of contributions describing their use for teaching general relativity will be extended in the future.

Light deflection in astronomy

How does the curvature of space or rather spacetime become apparent in astronomical observations? Though the deflection of light near the sun that was mentioned at the beginning appears to be negligibly small, it is yet noticeable in state-of-the-art observations.
A current astrometry mission, *Gaia* [6], is compiling the most comprehensive and most precise map of the Galaxy to date by measuring the positions of more than a billion stars with extremely high precision. Gaia will measure the positions of all objects brighter than

15th magnitude with a precision of 24 micro arc seconds. This precision corresponds to the diameter of a human hair at a distance of 1000 km.
The measurement accuracy is so high that light deflection due to the sun and even the larger planets must be taken into account. This is true not only for observations in directions close to the sun, but for *all* directions in the sky. As described above light is deflected by about 1.75 arc seconds when grazing the sun. When light is received from a direction perpendicular to the line of sight to the sun, it has been deflected by the sun by about 4000 micro arc seconds, light grazing the planet Jupiter by 16000 micro arc seconds. These angles of deflection are far larger than the accuracy of the Gaia telescope, so that without adjustment the results would be significantly distorted. Here, the general relativistic correction at first sight appears to be a necessary evil, however, the observations are also used to test Einstein's theory.

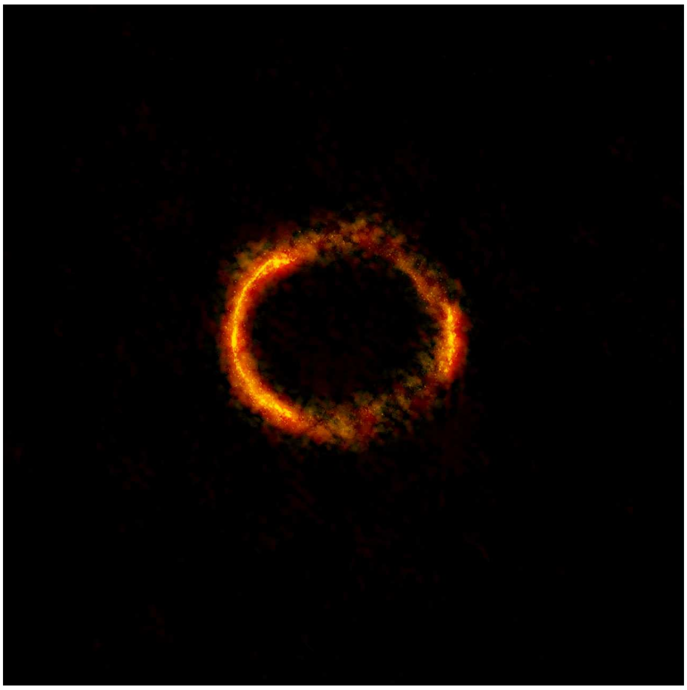


Abbildung 10: A nearly perfect Einstein ring, recorded with the ALMA radio telescope. Credit: ALMA (NRAO/ESO/NAOJ); B. Saxton NRAO/AUI/NSF

The image of the black hole in front of the Milky Way, shown at the beginning of this contribution, displays a ringlike lilac structure, a so-called *Einstein ring*. It is formed by light rays that originate from a lilac gas cloud just behind the black hole and that pass around the black hole equally on all sides due to the symmetry of the situation. Einstein predicted the formation of such a ring in 1936. Black holes and individual stars, however, are comparatively small so that it is very difficult to directly observe effects of the deflection of light in their vicinity. Einstein wrote: „Of course, there is no hope of observing this phenomenon directly.“ [7]. But if the mass that causes the light deflection is a whole galaxy or a cluster of galaxies, the deflection becomes observable. Fig. 10 shows a nearly perfect Einstein ring, recorded with the ALMA radio telescope [8]. The light originating from the galaxy SDP.81 shown in the picture travelled for 11.4 billion years to reach the earth; it stems from a time when the universe had 15% of its presentday size. The (invisible) foreground galaxy that acts as gravitational lens is at a distance of about 4 billion light years. The image of SDP.81 is strongly distorted, but it is also strongly magnified so that structures become visible that could not be detected without the gravitational lens. Here, gravitational light deflection can be used as an astronomical tool.

In the examples mentioned so far light is deflected by a small amount. Larger angles of deflection occur when light passes close to a neutron star or a black hole as shown for instance in Fig. 2. Because of the small size of these compact celestial bodies, it has not been possible so far to observe the effects of strong light deflection with a telescope. The Event Horizon Telescope [9] is a consortium of radio telescopes distributed over the whole surface of the earth. One of the goals is to obtain a detailed image of the black hole in the center of the Milky Way. Momentarily the resolution is not yet good enough, but extensions planned for the next years should render it possible to image the event horizon of the galactic black hole. It will then be possible, for the first time, to observe the influence of the strong gravitational field in the immediate vicinity of a black hole on the propagation of light.

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